Syrian Private University

# Algorithms & Data Structure I

Instructor: Dr. Mouhib Alnoukari

# 4 Algorithm Analysis

# Algorithm Analysis

Outline:

In this topic, we will examine code to determine the run time of various operations.

We will calculate the run times of:

- Operators — `+`, `-`, `=`, `+=`, `++`, etc.
- Control statements — `if`, `for`, `while`, `do-while`, `switch`
- Functions
- Recursive functions

# Motivation

The goal of algorithm analysis is to take a block of code and determine the asymptotic run time or asymptotic memory requirements based on various parameters

- Given an array of size $n$:
  - Selection sort requires $\Theta(n^2)$ time
  - Merge sort, quick sort, and heap sort all require $\Theta(n \ln(n))$ time
- However:
  - Merge sort requires $\Theta(n)$ additional memory
  - Quick sort requires $\Theta(\ln(n))$ additional memory
  - Heap sort requires $\Theta(1)$ memory

# Motivation

To properly investigate the determination of run times asymptotically:

- We will begin with machine instructions
- Discuss operations
- Control statements
  - Conditional statements and loops
- Functions
- Recursive functions

# Operators

Because each machine instruction can be executed in a fixed number of cycles, we may assume each operation requires a fixed number of cycles

- The time required for any operator is $\Theta(1)$ including:
  - Retrieving/storing variables from memory
  - Variable assignment `=`
  - Integer operations `+ - * / % ++ --`
  - Logical operations `&& || !`
  - Bitwise operations `& | ^ ~`
  - Relational operations `== != < <= => >`
  - Memory allocation and deallocation `new delete`

# Blocks of Operations

Each operation runs in $\Theta(1)$ time and therefore any fixed number of operations also run in $\Theta(1)$ time, for example:

```
// Swap variables a and b
int tmp = a;
a = b;
b = tmp;
```

# Blocks in Sequence

Suppose you have now analyzed a number of blocks of code run in sequence

```
template <typename T>
void update_capacity( int delta ) {
  T *array_old = array;
  int capacity_old = array_capacity;
  array_capacity += delta;
  array = new T[array_capacity];

  for ( int i = 0; i < capacity_old; ++i ) {
        array[i] = array_old[i];
  }

  delete[] array_old;
}
```

$\Theta(1)$

$\Theta(n)$

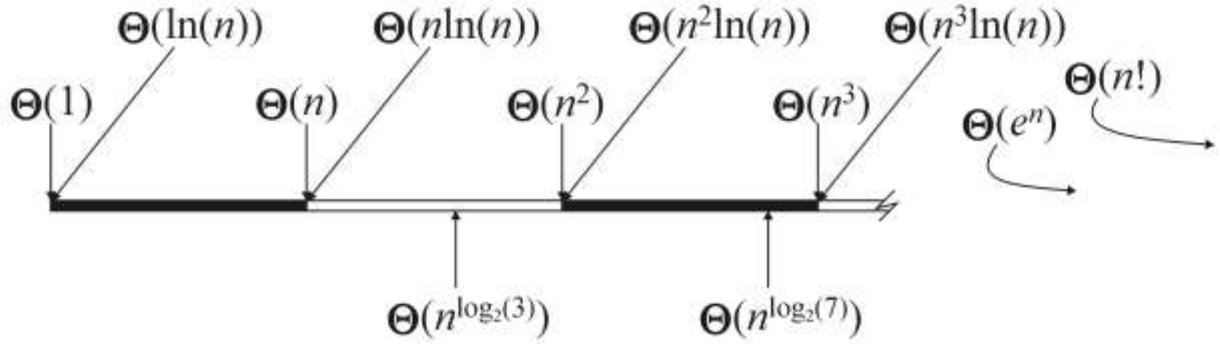
$\Theta(1)$

To calculate the total run time, add the entries: $\Theta(1 + n + 1) = \Theta(n)$

# Blocks in Sequence

Other examples include:

- Run three blocks of code which are $\Theta(1)$, $\Theta(n^2)$, and $\Theta(n)$
  - Total run time $\Theta(1 + n^2 + n) = \Theta(n^2)$
- Run two blocks of code which are $\Theta(n \ln(n))$, and $\Theta(n^{1.5})$
  - Total run time $\Theta(n \ln(n) + n^{1.5}) = \Theta(n^{1.5})$




- When considering a sum, take the dominant term

# Control Statements

Next we will look at the following control statements:

These are statements which potentially alter the execution of instructions

- Conditional statements
  - **`if`**, **`switch`**
- Condition-controlled loops
  - **`for`**, **`while`**, **`do-while`**
- Count-controlled loops
  - **`for i from 1 to 10 do ... end do;`**
- Collection-controlled loops
  - **`foreach ( int i in array ) { ... }      // C#`**

# Control Statements

Given any collection of nested control statements, it is always necessary to work inside out

– Determine the run times of the inner-most statements and work your way out

# Control Statements

Given:

```
if ( condition ) {
    // true body
} else {
    // false body
}
```

The run time of a conditional statement is:

- the run time of the condition (the test), plus
- the run time of the body which is run

In most cases, the run time of the condition is $\Theta(1)$

# Control Statements

In some cases, it is easy to determine which statement must be run:

```
int factorial ( int n ) {
        if ( n == 0 ) {
                return 1;
        } else {
                return n * factorial ( n – 1 );
        }
}
```

# Control Statements

In others, it is less obvious

– Find the maximum entry in an array:

```
int find_max( int *array, int n ) {
    max = array[0];

    for ( int i = 1; i < n; ++i ) {
        if ( array[i] > max ) {
            max = array[i];
        }
    }

    return max;
}
```

# Condition-controlled Loops

The for loop is a condition controlled statement:

```
for ( int i = 0; i < N; ++i ) {
        // ...
}
```

is identical to

```
int i = 0;                        // initialization
while ( i < N ) {                 // condition
    // ...
    ++i;                          // increment
}
```

# Condition-controlled Loops

The initialization, condition, and increment usually are single statements running in $\Theta(1)$

```
for ( int i = 0; i < N; ++i ) {
        // ...
}
```

# Condition-controlled Loops

The initialization, condition, and increment statements are usually $\Theta(1)$

For example,

```
for ( int i = 0; i < n; ++i ) {
    // ...
}
```

Assuming there are no break or return statements in the loop,
the run time is $\Theta(n)$

# Condition-controlled Loops

If the body does not depend on the variable (in this example, `i`), then the run time of :

```
for ( int i = 0; i < n; ++i ) {
    // code which is Theta(f(n))
}
```

is: $\Theta(n\,\mathrm{f}(n))$

If the body is $\Theta(\mathrm{f}(n))$, then the run time of the loop is $\Theta(n\,\mathrm{f}(n))$

# Condition-controlled Loops

For example,

```
int sum = 0;
for ( int i = 0; i < n; ++i ) {
    sum += 1;   //  Theta(1)
}
```

This code has run time:

$$\Theta(n \cdot 1) = \Theta(n)$$

# Condition-controlled Loops

Another example example,

```
int sum = 0;
for ( int i = 0; i < n; ++i ) {
    for ( int j = 0; j < n; ++j ) {
        sum += 1;     Theta(1)
    }
}
```

The previous example showed that the inner loop is $\Theta(n)$, thus the outer loop is:

$$\Theta(n \cdot n) = \Theta(n^2)$$

# Analysis of Repetition Statements

Suppose with each loop, we use a linear search an array of size $m$:

```
for ( int i = 0; i < n; ++i ) {
      // search through an array of size m
      // O( m );
}
```

The inner loop is $\mathbf{O}(m)$ and thus the outer loop is:

$$\mathbf{O}(n.m)$$

# Conditional Statements

## Consider this example

```
void Disjoint_sets::clear() {
    if ( sets == n ) {
        return;
    }

    max_height = 0;
    num_disjoint_sets = n;

    for ( int i = 0; i < n; ++i ) {
        parent[i] = i;
        tree_height[i] = 0;
    }
}
```

$\Theta(1)$

$\Theta(1)$

$\Theta(n)$

$\Theta(1)$

$$T_{\text{clear}}(n) = \begin{cases} \Theta(1) & sets = n \\ \Theta(n) & \text{otherwise} \end{cases}$$

# Analysis of Repetition Statements

If the body does depends on the variable (in this example, `i`), then the run time of:

```
for ( int i = 0; i < n; ++i ) {
    // code which is Theta(f(i,n))
}
```

is : $\boldsymbol{\Theta}\left(1 + \sum_{i=0}^{n-1} 1 + \mathrm{f}(i,n)\right)$ and if the body is

$\mathbf{O}(\mathrm{f}(i, n))$, the result is :

$$\mathbf{O}\left(1 + \sum_{i=0}^{n-1} 1 + \mathrm{f}(i,n)\right)$$

# Analysis of Repetition Statements

For example,

```
int sum = 0;
for ( int i = 0; i < n; ++i ) {
    for ( int j = 0; j < i; ++j ) {
        sum += i + j;
    }
}
```

The inner loop is $\mathbf{O}(1 + (1 + i)\ ) = \boldsymbol{\Theta}(i)$ hence the outer is:

$$\boldsymbol{\Theta}\left(1+\sum_{i=0}^{n-1} 1+i\right) = \boldsymbol{\Theta}\left(1+n+\sum_{i=0}^{n-1} i\right) = \boldsymbol{\Theta}\left(1+n+\frac{n(n-1)}{2}\right) = \boldsymbol{\Theta}\left(n^2\right)$$

# Analysis of Repetition Statements

As another example:

```
int sum = 0;
for ( int i = 0; i < n; ++i ) {
    for ( int j = 0; j < i; ++j ) {
        for ( int k = 0; k < j; ++k ) {
            sum += i + j + k;
        }
    }
}
```

From inside to out:

$\Theta(1)$

$\Theta(j)$

$\Theta(i^2)$

$\Theta(n^3)$

# Control Statements

Switch statements appear to be nested if statements:

```
switch( i ) {
     case 1:    /* do stuff */ break;
     case 2:    /* do other stuff */ break;
     case 3:    /* do even more stuff */ break;
     case 4:    /* well, do stuff */ break;
     case 5:    /* tired yet? */ break;
     default:   /* do default stuff */
}
```

# Control Statements

Thus, a switch statement would appear to run in $\mathbf{O}(n)$ time where $n$ is the number of cases, the same as nested if statements

– Why then not use:

```
if ( i == 1 ) { /* do stuff */ }
else if ( i == 2 ) { /* do other stuff */ }
else if ( i == 3 ) { /* do even more stuff */ }
else if ( i == 4 ) { /* well, do stuff */ }
else if ( i == 5 ) { /* tired yet? */ }
else { /* do default stuff */ }
```

# Serial Statements

Suppose we run one block of code followed by another block of code.

Such code is said to be run *serially*

If the first block of code is $\mathbf{O}(\mathrm{f}(n))$ and the second is $\mathbf{O}(\mathrm{g}(n))$, then the run time of two blocks of code is:

$$\mathbf{O}(\ \mathrm{f}(n) + \mathrm{g}(n)\ )$$

which usually (for algorithms not including function calls) simplifies to one or the other.

# Serial Statements

Consider the following two problems:

- search through a random list of size $n$ to find the maximum entry, and
- search through a random list of size $n$ to find if it contains a particular entry

What is the proper means of describing the run time of these two algorithms?

# Serial Statements

Searching for the maximum entry requires that each element in the array be examined, thus, it must run in $\mathbf{\Theta}(n)$ time.

Searching for a particular entry may end earlier: for example, the first entry we are searching for may be the one we are looking for, thus, it runs in $\mathbf{O}(n)$ time.

# Serial Statements

Therefore:

– if the leading term is big-$\mathbf{\Theta}$, then the result must be big-$\mathbf{\Theta}$, otherwise
– if the leading term is big-$\mathbf{O}$, we can say the result is big-$\mathbf{O}$

For example,

$$\mathbf{O}(n) + \mathbf{O}(n^2) + \mathbf{O}(n^4) = \mathbf{O}(n + n^2 + n^4) = \mathbf{O}(n^4)$$

$$\mathbf{O}(n) + \mathbf{\Theta}(n^2) = \mathbf{\Theta}(n^2)$$

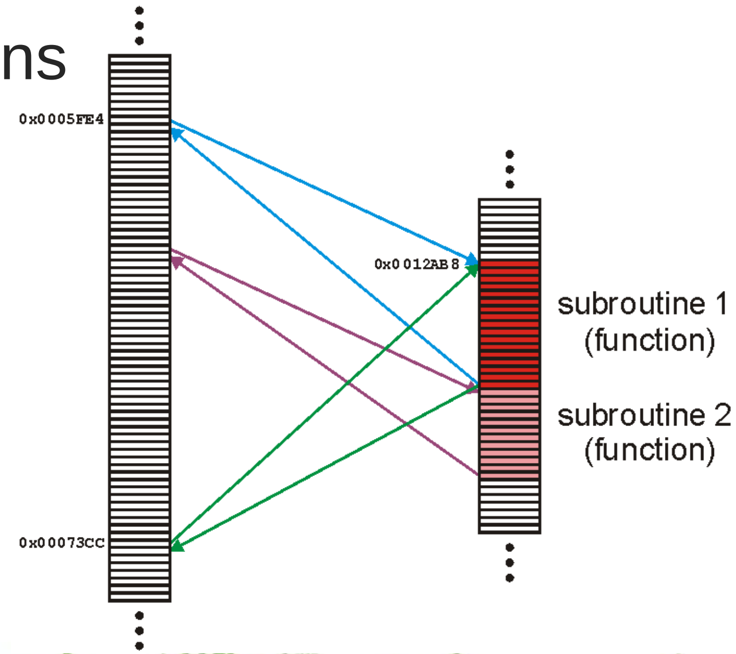
$$\mathbf{O}(n^2) + \mathbf{\Theta}(n) = \mathbf{O}(n^2)$$

$$\mathbf{O}(n^2) + \mathbf{\Theta}(n^2) = \mathbf{\Theta}(n^2)$$

# Functions

A function (or subroutine) is code which has been separated out, either to:

- and repeated operations
  - e.g., mathematical functions
- group related tasks
  - e.g., initialization



# Functions

Because a subroutine (function) can be called from anywhere, we must:

- prepare the appropriate environment
- deal with arguments (parameters)
- jump to the subroutine
- execute the subroutine
- deal with the return value
- clean up

# Functions

Fortunately, this is such a common task that all modern processors have instructions that perform most of these steps in one instruction.

Thus, we will assume that the overhead required to make a function call and to return is $\Theta(1)$.

# Functions

Because any function requires the overhead of a function call and return, we will always assume that

$$T_f = \Omega(1)$$

That is, it is impossible for any function call to have a zero run time.

# Functions

Thus, given a function $\mathrm{f}(n)$ (the run time of which depends on $n$) we will associate the run time of $\mathrm{f}(n)$ by some function $\mathrm{T}_{\mathrm{f}}(n)$

– We may write this to $\mathrm{T}(n)$

Because the run time of any function is at least $\mathbf{O}(1)$, we will include the time required to both call and return from the function in the run time.

# Functions

Consider this function:

```
void Disjoint_sets::set_union( int m, int n ) {
        m = find( m );
        n = find( n );

        if ( m == n ) {
                return;
        }

        --num_disjoint_sets;

        if ( tree_height[m] >= tree_height[n] ) {
            parent[n] = m;

            if ( tree_height[m] == tree_height[n] ) {
                ++( tree_height[m] );
                max_height = std::max( max_height, tree_height[m] );
            }
        } else {
            parent[m] = n;
        }
}
```

$2T_{\text{find}}$

$T_{\text{set\_union}} = 2T_{\text{find}} + \Theta(1)$

$\Theta(1)$

# Recursive Functions

A function is relatively simple (and boring) if it simply performs operations and calls other functions.

Most interesting functions designed to solve problems usually end up calling themselves.

- Such a function is said to be *recursive*

# Recursive Functions

As an example, we could implement the factorial function recursively:

```
int factorial( int n ) {
    if ( n <= 1 ) {
        return 1;
    } else {
        return n * factorial( n - 1 );
    }
}
```

$$\Theta(1)$$

$$T_{!}(n-1)+\Theta(1)$$

# Recursive Functions

Thus, we may analyze the run time of this function as follows:

$$\mathrm{T}_{!}(n) = \begin{cases} \boldsymbol{\Theta}(1) & n \le 1 \\ \mathrm{T}_{!}(n-1) + \boldsymbol{\Theta}(1) & n > 1 \end{cases}$$

We don’t have to worry about the time of the conditional ($\boldsymbol{\Theta}(1)$) nor is there a probability involved with the conditional statement.

# Recursive Functions

The analysis of the run time of this function yields a recurrence relation:

$$\mathrm{T}_!(n) = \mathrm{T}_!(n-1) + \boldsymbol{\Theta}(1) \qquad \mathrm{T}_!(1) = \boldsymbol{\Theta}(1)$$

This recurrence relation has Landau symbols…

– Replace each Landau symbol with a representative function:

$$\mathrm{T}_!(n) = \mathrm{T}_!(n-1) + 1 \qquad \mathrm{T}_!(1) = 1$$

# Recursive Functions

We can examine the first few steps:

$$\begin{aligned} T_!(n) &= T_!(n-1) + 1 \\ &= T_!(n-2) + 1 + 1 = T_!(n-2) + 2 \\ &= T_!(n-3) + 3 \end{aligned}$$

From this, we see a pattern:

$$T_!(n) = T_!(n-k) + k$$

# Recursive Functions

If $k = n - 1$ then:

$$
\begin{aligned}
T_!(n) &= T_!(n - (n - 1)) + n - 1 \\
&= T_!(1) + n - 1 \\
&= 1 + n - 1 = n
\end{aligned}
$$

Thus, $T_!(n) = \Theta(n)$

# Recursive Functions

Analyzing the function, we get:

```
void sort( int * array, int n ) {
    if ( n <= 1 ) {
        return;
    }

    int posn = 0;
    int max = array[posn];

    for ( int i = 1; i < n; ++i ) {
        if ( array[i] > max ) {
            posn = i;
            max = array[posn];
        }
    }

    int tmp = array[n - 1];
    array[n - 1] = array[posn];
    array[posn] = tmp;

    sort( array, n - 1 );
}
```

$\mathrm{T}(0) = \mathrm{T}(1) = \Theta(1)$

$\Theta(1)$

$\Theta(1)$ $\Theta(n)$

$\Theta(1)$

$\mathrm{T}(n-1)$

$$\begin{aligned}\mathrm{T}(n) &= \Theta(1) + \Theta(n) + \Theta(1) + \mathrm{T}(n-1)\\ &= \mathrm{T}(n-1) + \Theta(n)\end{aligned}$$

# Recursive Functions

Thus, replacing each Landau symbol with a representative, we are required to solve the recurrence relation:

$$\mathrm{T}(n) = \mathrm{T}(n - 1) + n \qquad \mathrm{T}(1) = 1$$

$$-1 - n + (n + 1)\left(\frac{n}{2} + 1\right)$$

$$\frac{1}{2}n + \frac{1}{2}n^2$$

# Recursive Functions

Consequently, the sorting routine has the run time

$$\mathrm{T}(n) = \Theta(n^2)$$

To see this by hand, consider the following

$$\begin{aligned}
\mathrm{T}(n) &= \mathrm{T}(n-1) + n \\
&= \big(\mathrm{T}(n-2) + (n-1)\big) + n \\
&= \mathrm{T}(n-2) + n + (n-1) \\
&= \mathrm{T}(n-3) + n + (n-1) + (n-2) \\
&\ \vdots \\
&= \mathrm{T}(1) + \sum_{i=2}^{n} i = 1 + \sum_{i=2}^{n} i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\end{aligned}$$

# Recursive Functions

Consider, instead, a binary search of a sorted list:

- Check the middle entry
- If we do not find it, check either the left- or right-hand side, as appropriate

Thus, $\mathrm{T}(n) = \mathrm{T}((n - 1)/2) + \Theta(1)$

# Recursive Functions

Also, if $n = 1$, then $\mathrm{T}(1) = \Theta(1)$

Thus we have to solve:

$$\mathrm{T}(n) = \begin{cases} 1 & n = 1 \\ \mathrm{T}\left(\dfrac{n-1}{2}\right) + 1 & n > 1 \end{cases}$$

Solving this can be difficult, in general, so we will consider only special values of $n$.

Assume $n = 2^k - 1$ where $k$ is an integer

Then $(n - 1)/2 = (2^k - 1 - 1)/2 = 2^{k-1} - 1$

# Recursive Functions

For example, searching a list of size 31 requires us to check the center.

If it is not found, we must check one of the two halves, each of which is size 15

$$31 = 2^5 - 1$$

$$15 = 2^4 - 1$$

# Recursive Functions

Thus, we can write:

$$
\begin{aligned}
\mathrm{T}(n) &= \mathrm{T}(2^k - 1) \\
&= \mathrm{T}\left(\frac{2^k - 1 - 1}{2}\right) + 1 \\
&= \mathrm{T}(2^{k-1} - 1) + 1 \\
&= \mathrm{T}\left(\frac{2^{k-1} - 1 - 1}{2}\right) + 1 + 1 \\
&= \mathrm{T}(2^{k-2} - 1) + 2 \\
&\vdots
\end{aligned}
$$

# Recursive Functions

Notice the pattern with one more step:

$$= T(2^{k-1} - 1) + 1$$

$$= T\left(\frac{2^{k-1} - 1 - 1}{2}\right) + 1 + 1$$

$$= T(2^{k-2} - 1) + 2$$

$$= T(2^{k-3} - 1) + 3$$

$$\vdots$$

# Recursive Functions

Thus, in general, we may deduce that after $k - 1$ steps:

$$\begin{aligned} \mathrm{T}(n) &= \mathrm{T}(2^k - 1) \\ &= \mathrm{T}(2^{k-(k-1)} - 1) + k - 1 \\ &= \mathrm{T}(1) + k - 1 = k \end{aligned}$$

because $\mathrm{T}(1) = 1$

# Recursive Functions

Thus, $T(n) = k$, but $n = 2^k - 1$
Therefore $k = \lg(n + 1)$

However, recall that $f(n) = \Theta(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ for $0 < c < \infty$

$$\lim_{n\to\infty} \frac{\lg(n+1)}{\ln(n)} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)\ln(2)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{(n+1)\ln(2)} = \frac{1}{\ln(2)}$$

Thus, $T(n) = \Theta(\lg(n + 1)) = \Theta(\ln(n))$

# Cases

As well as determining the run time of an algorithm, because the data may not be deterministic, we may be interested in:

- Best-case run time
- Average-case run time
- Worst-case run time

In many cases, these will be significantly different.

# Cases

Searching a list linearly is simple enough

We will count the number of comparisons

- Best case:
  - The first element is the one we're looking for: $\mathbf{O}(1)$
- Worst case:
  - The last element is the one we're looking for, or it is not in the list: $\mathbf{O}(n)$
- Average case?
  - We need some information about the list...

# Cases

Assume the case we are looking for is in the list and equally likely distributed.

If the list is of size $n$, then there is a $1/n$ chance of it being in the $i$th location

Thus, we sum:

$$\frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}$$

which is $\mathbf{O}(n)$.

# Exercise

Prove that running time $T(n) = n^3 + 20n + 1$ is $O(n^3)$

**Proof:** by the Big-Oh definition, $T(n)$ is $O(n^3)$ if $T(n) \leq c \cdot n^3$ for some $n \geq n_0$. Let us check this condition: if $n^3 + 20n + 1 \leq c \cdot n^3$ then $1 + \frac{20}{n^2} + \frac{1}{n^3} \leq c$. Therefore, the Big-Oh condition holds for $n \geq n_0 = 1$ and $c \geq 22$ ($= 1 + 20 + 1$). Larger values of $n_0$ result in smaller factors $c$ (e.g., for $n_0 = 10$ $c \geq 1.201$ and so on) but in any case the above statement is valid.

# Exercise

Prove that running time $T(n) = n^3 + 20n + 1$ is not $O(n^2)$

**Proof:** by the Big-Oh definition, $T(n)$ is $O(n^2)$ if $T(n) \leq c \cdot n^2$ for some $n \geq n_0$ . Let us check this condition: if $n^3 + 20n + 1 \leq c \cdot n^2$ then $n + \frac{20}{n} + \frac{1}{n^2} \leq c$ . Therefore, the Big-Oh condition cannot hold (the left side of the latter inequality is growing infinitely, so that there is no such constant factor $c$).